PH 5840 Lecture 17

Note Title 9/21/2015

Quantum circuit Model.

① Single qubit operations :-

Qubit : $|\psi\rangle = a|0\rangle + b|1\rangle \in \mathbb{C}^2$ (with $|a|^2+|b|^2=1$)

Any single-qubit gate must map unit vectors in $\mathbb{C}^2 \longrightarrow$ unit vectors in $\mathbb{C}^2$

$\Rightarrow$ 2×2 unitary matrices : elements of SU(2)

(a) Basic single-qubit gates :-

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\sigma_x + \sigma_z)$$

"Phase" gate : $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

"$\pi/8$" gate : $G_{\pi/8} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8} \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}$

(b) "Rotation" operators :-

$$R_x(\theta) = e^{-i\theta\frac{\sigma_x}{2}} = \left(\cos\frac{\theta}{2}\right) I - \left(i\sin\frac{\theta}{2}\right)\sigma_x$$

$$R_y(\theta) = e^{-i\theta\sigma_y/2} = \left(\cos\frac{\theta}{2}\right) I - \left(i\sin\frac{\theta}{2}\right)\sigma_y$$

$$R_z(\theta) = e^{-i\theta\sigma_z/2} = \left(\cos\frac{\theta}{2}\right) I - \left(i\sin\frac{\theta}{2}\right)\sigma_z$$

$$R_x(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

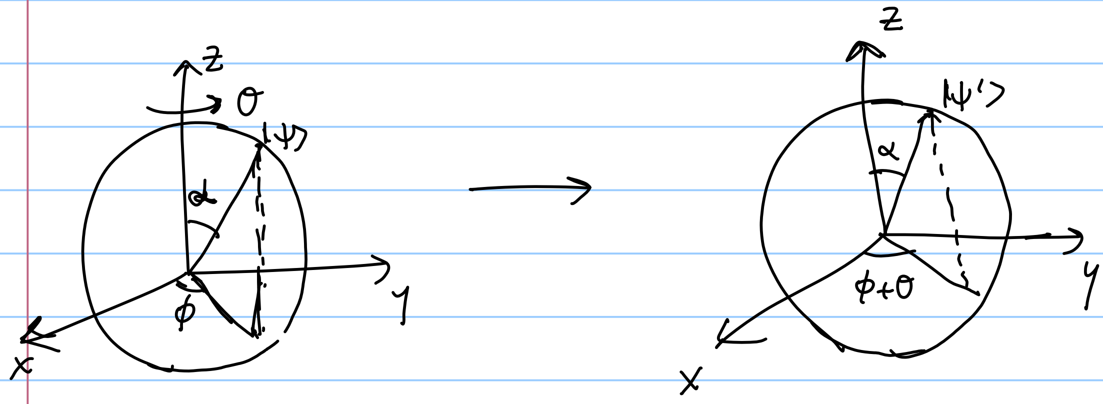
$$R_y(\theta) = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

Recall: $|\psi\rangle = \cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\alpha}{2}\right)|1\rangle$

$\therefore$ $R_z(\theta)$ rotates $|\psi\rangle$ by an angle θ about z-axis!

$$R_z(\theta)|\psi\rangle = |\psi'\rangle = e^{-i\theta/2}\left(\cos\frac{\alpha}{2}|0\rangle + e^{i(\phi+\theta)}\sin\frac{\alpha}{2}|1\rangle\right)$$



$\|$ly $R_x(\theta)$, $R_y(\theta)$: rotations by θ about the x and y axes.

Thm 1 :- Z-Y decomposition of a single qubit:-

Given $U \in SU(2)$, $\exists\ \alpha, \beta, \gamma, \delta$ (all reals)

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

"Proof": Rows and columns of U are orthonormal.

$$|u_1\rangle = e^{-i(\lambda_1+\lambda_2)} \cos\gamma/2 \; |0\rangle + e^{+i(\lambda_1-\lambda_2)} \sin\gamma/2 \, |1\rangle$$

$$|u_1^\perp\rangle = -e^{i(-\lambda_1+\lambda_2)} \sin\gamma/2 \, |0\rangle + e^{i(\lambda_1+\lambda_2)} \cos\gamma/2 \; |1\rangle$$

$$U = e^{i\alpha} \begin{pmatrix} e^{-i(\lambda_1+\lambda_2)} \cos\gamma/2 & -e^{i(-\lambda_1+\lambda_2)} \sin\gamma/2 \\ e^{i(\lambda_1-\lambda_2)} \sin\gamma/2 & e^{i(\lambda_1+\lambda_2)} \cos\gamma/2 \end{pmatrix}$$

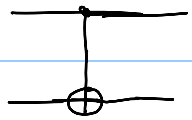
Pick $\lambda_1 = \frac{\beta}{2}$, $\lambda_2 = \frac{\delta}{2}$

$$\therefore U = e^{i\alpha} \begin{pmatrix} e^{-i\left(\frac{\beta}{2}+\frac{\delta}{2}\right)} \cos\gamma/2 & -e^{i\left(-\frac{\beta}{2}+\frac{\delta}{2}\right)} \sin\frac{\gamma}{2} \\ e^{i(\beta/2-\delta/2)} \sin\gamma/2 & e^{i(\beta/2+\delta/2)} \cos\gamma/2 \end{pmatrix}$$

$$\equiv e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) \,//.$$

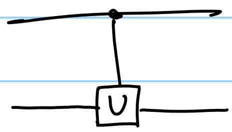
QED!

(c) Controlled operations :-

CNOT :  $\equiv \Lambda(\sigma_x)$ (Controlled NOT)

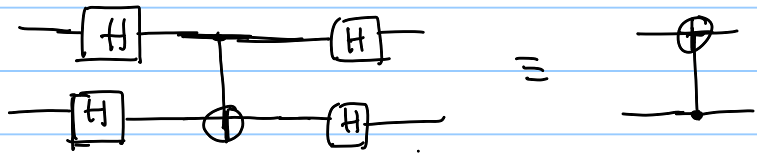
More generally: $\Lambda(U) \; : \; |c\rangle|t\rangle \rightarrow |c\rangle U^c |t\rangle$

$$= (I \otimes U)|c\rangle|t\rangle$$

(controlled-unitary) 

* Circuit identities

(a)
$$H\sigma_x H = \sigma_z$$
$$H\sigma_y H = -\sigma_y$$
$$H(\sigma_z)H = \sigma_x$$

(b) 

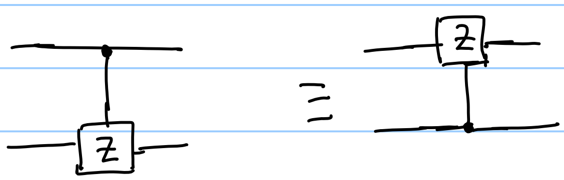
$|0\rangle|0\rangle \rightarrow (\text{CNOT})(|+\rangle|+\rangle) = |+\rangle|+\rangle$ $\qquad$ $|0\rangle|0\rangle$

$|1\rangle|0\rangle \rightarrow (\text{CNOT})(|-\rangle|+\rangle) = |-\rangle|+\rangle$ $\quad \xrightarrow{H \otimes H} \quad$ $|1\rangle|0\rangle$

$|0\rangle|1\rangle \rightarrow (\text{CNOT})(|+\rangle|-\rangle) = |-\rangle|-\rangle$ $\qquad$ $|1\rangle|1\rangle$

$|1\rangle|1\rangle \rightarrow (\text{CNOT})(|-\rangle|-\rangle) = |+\rangle|-\rangle$ $\qquad$ $|0\rangle|1\rangle$

$\Downarrow$ Flips phase of first qubit if second qubit is $|-\rangle$!

(c) 

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$(\Lambda(\sigma_z))$